理學碩士學位論文

UNIVERSALITY OF $k \cdot 3^n$ - AND $k \cdot 4^n$ - CASCADES FOR AREA-PRESERVING MAPS

서울大學校 大學院

物理學科

김 상 윤

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UNIVERSALITY OF $k \cdot 3^n$ - AND $k \cdot 4^n$ - CASCADES FOR AREA-PRESERVING MAPS

UNDER THE SUPERVISION OF
PROF. KOO-CHUL LEE

BY
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ABSTRACT

We have studied numerically period-trebling and period-quardrupling ($k \cdot 3^n$, $k \cdot 4^n$) cascades of periodic orbits of two dimensional area-preserving maps. Period-trebling δ_n-sequence converges as $n \to \infty$, and the limit value is 20.2. Unlike the period-doubling cascades, each of period-trebling α_n-and β_n-sequence converges alternately, and two limit values of α_n-sequence are α_1 (= -17.9) and α_2 (= 2.45) and two limit values of β_n-sequence are β_1 (= -31.0) and β_2 (= 6.02). The structure of periodic orbits reproduce itself asymtotically from one 1/3-resonance to every other 1/3-resonance under the rescaling and the rescaling factors $\alpha (= \alpha_1 \cdot \alpha_2)$ and $\beta (= \beta_1 \cdot \beta_2)$ are -44.0 and -187. Period-quadrupling sequence confirm the universal limiting behavior and the universal constants δ, α and β are 24.5, -5.61 and 14.3.

I. INTRODUCTION

There has been interest in the transition from the regular motion to the irregular motion in the dynamical systems. It is generally believed that two dimensional area-preserving maps have generic properties of ergodic motion and one dimensional noninvertible and higher dimensional area-contracting maps have generic properties of turbulent motion. For the two dimensional area-preserving maps, KAM theorem says that when a non-integrable canonical perturbation is acted on an integrable mapping, invariant circles with sufficiently irrational winding numbers are preserved, albeit in distorted form, while invariant circles with rational and nearly rational winding numbers are destroyed and the measure of the destroyed region is, though small, not zero. But KAM theorem does not say what happens to the motion in the destroyed region. By the Poincaré -Birkhoff theorem, any invariant circle of period n breaks up into many pairs of elliptical and ordinary hyperbolic orbits of period n when a nonintergrable canonical perturbation is acted. As the perturbation is increased, at ℓ/m - resonance (m and $\ell \geq 5$, relatively prime) a pair of elliptical and ordinary hyperbolic orbits of period m times the original period are born around the original elliptical orbit born in consequence of the Poincaré-Birkhoff theorem and finally at 1/2-resonance (bifurcation) a new elliptical orbit of the doubled period is born around the original orbit which now turns into the inversion hyperbolic orbit.[1-2] The newly born elliptical orbit either by resonance or bifurcation is now the basis of the above process, and this process repeat infinite times. The stable separatrix and the unstable separatrix emanating from the fixed hyperbolic point or two hyperbolic points of the same unstable orbit intersect each other infinitely to form a kind of network with infinitely tight loops. Therefore, near the separatrices of the unstable orbit a chaotic region is formed. In this region, another unstable orbit is born, separatrices of two different unstable orbits intersect each other infinitely and the chaotic regions are broadened. This phenomenon is called the resonance overlap which is the criterion of the stochasticity in the theory of the nonlinear oscillation developed by Chirikov and Zaslabski.[3-4] As the KAM torus encloses this chaotic region for N (degree of freedom)=2, this chaotic region is a locally unstable region. Because KAM torus does not enclose the locally unstable region for $N \geq 3$, the locally chaotic regions are connected to form a globally chaotic region. This phenomenon is called the Arnold diffusion.[2,5]

In recent years, Feigenbaum's discovery of the universal scaling behavior of the period-doubling cascade of the one dimensional noninvertible maps expedited the study of the period-doubling cascade of the two dimensional area-preserving maps.[6] The universal scaling behavior has been discovered by the numerical study and the renormalization method.[5-20] By the resonance, there are in general $k \cdot r^n$ $(r \geq 3)$ cascades in the Hamiltonian maps. Therefore it would be interesting to study the $k \cdot r^n$ $(r \geq 3)$ cascades.

II. MULTIFURCATION FOR THE 2-DIM. REVERSIBLE AREA-PRESERVING MAPS

We use the following form for the 2-dim. reversible area-preserving maps,

$$T: X_{n+1} = -Y_n + 2h(X_n), \ Y_{n+1} = X_n; \ h(X) = (1-aX^2)/2$$

Most of different forms of maps studied in literature are all equivalent to the above form. Since T is a reversible map, $T = I_2 \cdot I_1$; $I_1^2 = I_2^2 = 1$,

$$I_1: \ X_{n+1} = X_n, \ Y_{n+1} = -Y_n + 2h(X_n)$$

$$I_2: \ X_{n+1} = Y_n, \ Y_{n+1} = X_n$$

The set of the invariant points under the operation of I_1 or I_2 forms a line and we call it a symmetry line. It can be easily shown that two or no points of every orbit of even period and one or no points of every orbit of odd period lie on any given symmetry line. It is of great advantage to use the reversibility for the numerical work. The symmetry lines of T are Y=X and Y=h(X).

A quantity R called the residue makes the study of the behavior of the neighborhood around a periodic orbit effective. The residue is given R = (2-TrM)/4, Where M is a Jacobian matrix of T^n about an orbit of period n.

The periodic orbit is stable for $0<R<1$ (except for $R = 3/4$ and sometimes $1/2$), and unstable for $R<0$ and $R>1$. In the stable case, nearby points to a periodic point move around it in ellipses under M at rate α rotations/period given by $R = \sin^2(\alpha/2)$. Therefore the orbit is called an elliptical orbit. In the unstable case nearby points move on hyperbolae, alternating between corresponding branches if $R>1$ (inversion hyperbolic orbit), and staying on one branch if $R<0$ (ordinary hyperbolic orbit). In the special cases $R=0$, 1, 3/4, 1/2 corresponding to the low order resonances, M is not sufficient to describe the behavior of nearby points.

When the residue R of a stable orbit passes the value $\sin^2(\pi\ell/m)$, as the nonintergrable parameter a changes, where ℓ and m are coprime, $m\geq5$ and $m>\ell>0$, a pair of stable and unstable orbits of period m times the original period are born near the original orbit. The resonances of order 3 ($m=3$) and order 4 ($m=4$) are exceptional.

For the generic bifurcation ($R=1$), a new elliptical orbit of doubled period is born around the original orbit which turns to the inversion hyperbolic orbit. When $R=0$, two new elliptical orbits of the same period are bifurcated from the original orbit which turns to the ordinary hyperbolic orbit. When R($\geq$1) passes to 1, a new ordinary hyperbolic orbit of doubled period is bifurcated from the original inversion hyperbolic orbit which turns to a elliptical orbit. As the nonintergrable parameter a is further increased, the above elliptic orbit turns to the ordinary hyperbolic orbit.

Before the residue R passes the resonance value (R=3/4), a pair of stable and unstable orbits of period 3 times the original period are born, at the 1/3-resonance value the newly born unstable orbit is absorbed by the original orbit, and after the residue R passes the 1/3-resonance value, the original periodic orbit emits the newly born unstable orbit. (Fig. 1, Fig. 2, Fig. 3)

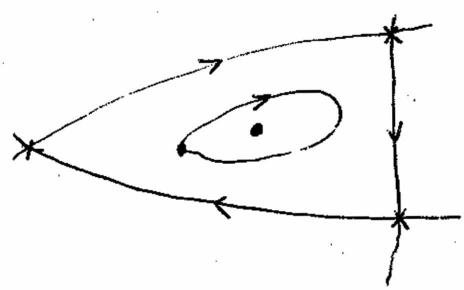
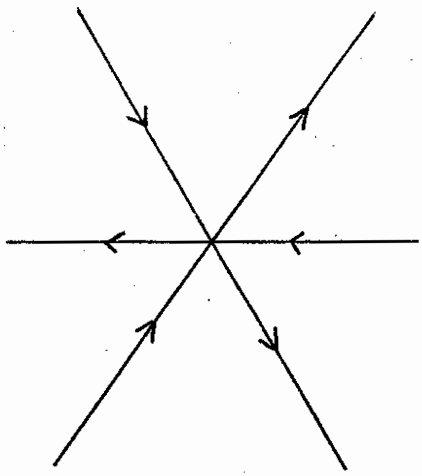


Fig. 1. Phase flows under T^3 when R<3/4. ● denotes an elliptic point and x denotes an ordinary hyperbolic point of period 3.



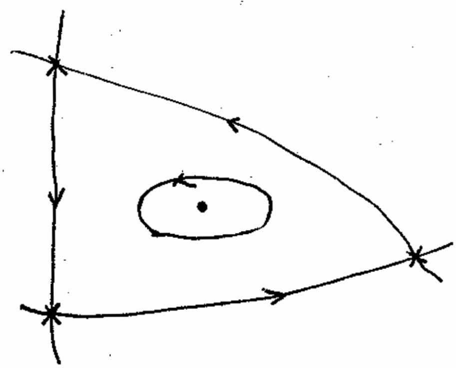


Fig. 2. Phase flow under T^3 when R=3/4

Fig. 3. Phase flow under T^3 when R>3/4

When R=3/4, the original elliptic orbit is unstable (Fig. 2).

For the 1/4-resonance, there are two cases. One case is that at the resonance value(R=1/2), a pair of elliptic and ordinary hyperbolic orbit are born. The other case is the same as the case of the 1/3-resonance (Fig. 4, Fig. 5, Fig. 6)

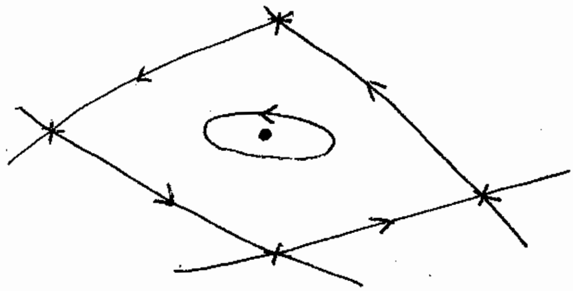


Fig. 4. Phase flow under T^{16} when $R<1/2$.

● denotes an elliptic orbit of period 4.

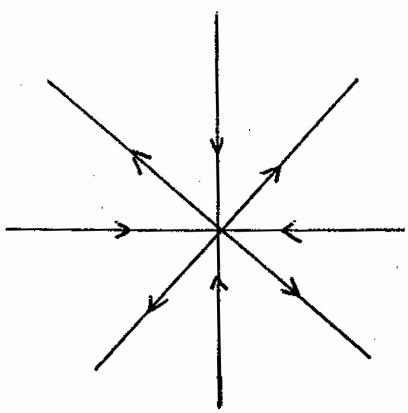


Fig. 5. Phase flow unter T^{16} when $R=1/2$

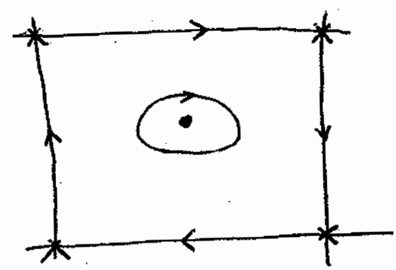


Fig. 6. Phase flow under T^{16} when $R>1/2$

At the resonance, the original orbit is unstable (Fig. 5).

III. PERIOD-TREBLING AND PERIOD-QUADRUPLING CASCADES

We use the following form for the 2-dim. reversible area-preserving maps,

$$T: \quad X_{n+1} = -Y_n + 2h(X_n), \quad Y_{n+1} = X_n,$$

$$\text{where } h(X) = (1-aX^2)/2 \qquad (1)$$

Since T is a reversible map, $T = I_2 \cdot I_1$; $I_1^2 = I_2^2 = 1$,

$$I_1: \quad X_{n+1} = Y_n, \quad Y_{n+1} = -Y_n + 2h(X_n) \qquad (2)$$

$$I_2: \quad X_{n+1} = Y_n, \quad Y_{n+1} = X_n \qquad (3)$$

The symmetry lines of T are Y=X and Y=h(X).

Before the residue R passes the 1/3-resonance value (R=3/4), a pair of stable and unstable orbits of period 3 times the original period are born, at the resonance value the newly born unstable orbit is absorbed by the original periodic orbit, and after the residue R passes the 1/3-resonance value, the original periodic orbit emits the newly born unstable orbit.

Let us see the structure of the newly born orbit by the resonance of order 3. One point of the orbits of period 3^n lies on the symmetry line Y=X, and another point lies on the Y=h(X). There are two cases for the $k \cdot 3^n$-cascades, where k is even. One case is that two different points of the orbits of period $k\cdot 3^n$ lie on the Y=h(X), and the other case is that two different points of the orbits of period $k\cdot 3^n$ lie on the Y=X. For example, the $2\cdot 3^n$-cascade is the former case and the $6\cdot 3^n$-cascade is the latter case, where the basic orbit of period 6 is the orbit bifurcated from the orbit of period 3. For the 3^n-cascade, let us call the point which lies on the Y=X the initial point. The initial point Z_0, the 1/3-way point $Z_{3^{n-1}}$ and the 2/3-way point $Z_{2\cdot 3^{n-1}}$ of the orbit of period 3^n enclose the initial point of the orbit of period 3^{n-1}, and the 1/6-way point $Z_{(3^{n-1}-1)/2}$, the 1/2-way point $Z_{(3^n-1)/2}$ and the 5/6-way point $Z_{(5\cdot 3^{n-1}-1)/2}$ enclose the 1/2-way point of the orbit of period 3^{n-1}; $Z_\tau=(X_\tau, Y_\tau)$.

An orbit of odd period (2m+1) with the initial point on the Y=X satisfies

$$X_{m+\ell} = X_{m-\ell},\ Y_{m+\ell+1} = Y_{m-\ell+1},\ \ell = 0.\ 1,\ \ldots \qquad (4)$$

Since $Y_\tau = X_{\tau+1}$, $X_{2\cdot 3^{n-1}} = Y_{2\cdot 3^{n-1}+1}$ and $X_{3^{n-1}}=Y_{3^{n-1}+1}$

By (4), $Y_{2\cdot 3^{n-1}+1} = Y_{3^{n-1}}$ and $Y_{3^{n-1}+1} = Y_{2\cdot 3^{n-1}}$. Therefore the 2/3-way point $Z_{2\cdot 3^{n-1}}$ is the reflection point of the 1/3-way point $Z_{3^{n-1}}$ about the Y = X.

By (4), $X_{(3^{n-1}-1)/2} = X_{(5\cdot 3^{n-1}-1)/2}$.

Hence the X-components of the 1/6-way point and the 5/6-way point are equal. The intersection point between the Y = X line and the line which joins the 1/3-way point and the 2/3-way point is $Z_{0,c}$ which is $[(X_{3^{n-1}} + Y_{3^{n-1}})/2, (X_{3^{n-1}} + Y_{3^{n-1}})/2]$, and the intersection point betwe the Y = h(X) line and the line which joins the 1/6-way point and the 5/6-way point is $Z_{1/2,c}$, which is $[X_{(3^{n-1}-1)}$ $h(X_{(3^{n-1}-1)/2})]$. Two different points of the orbit of period $2\cdot 3^n$ lie on the Y = h(X) line. Let us call on point which is left to the other point the initial point By (4) $X_{2\cdot 3^{n-1}} = X_{4\cdot 3^{n-1}}$ and $X_{3^{n-1}} = X_{5\cdot 3^{n-1}}$. Hence the X-components of the 1/3-way point $Z_{2\cdot 3^{n-1}}$ and 2/3-way point $Z_{4\cdot 3^{n-1}}$ are equal, and so are the X-components of the 1/6-way point $Z_{3^{n-1}}$ and 5/6-way point $Z_{5\cdot 3^{n-1}}$. $Z_{0,c}$ is $[X_{2\cdot 3^{n-1}}, h(X_{2\cdot 3^{n-1}})]$ and $Z_{1/2,c}$ is $[X_{3^{n-1}}, h(X_{3^{n-1}})]$, where $Z_{0,c}$ is the intersection point between Y=h(X) line and the line which joins the 1/3-way point and 2/3-way point. $Z_{1/2,c}$ is the point defined for the 3^n-cascade.

Two different points of the orbit of period $6\cdot 3^n$ lie on t Y = X line, where the basic orbit of period 6 is the orbit bifurcated from the orbit of period 3. When the

1/2-way point of the orbit of period 2m lie on Y = X,

$$X_{m+\ell} = X_{m-\ell-1} \quad \text{and} \quad Y_{m+\ell+1} = Y_{m-\ell},\ \ell = 0,1,2,\ldots \quad (5)$$

By(5), $X_{6\cdot 3^{n-1}} = Y_{12\cdot 3^{n-1}}$ and $X_{12\cdot 3^{n-1}} = Y_{6\cdot 3^n}$. Therefore the 1/3-way point $Z_{6\cdot 3^{n-1}}$ is the reflection point of the 2/3-way point $Z_{12\cdot 3^{n-1}}$ about the Y=X line, and the 1/6-way point $Z_{3\cdot 3^{n-1}}$ is the reflection point of the 5/6-way point $Z_{15\cdot 3^{n-1}}$ about the Y = X line. $Z_{0,c}$ is $[(X_{6\cdot 3^{n-1}} + Y_{6\cdot 3^{n-1}})/2, (X_{6\cdot 3^{n-1}} + Y_{6\cdot 3^{n-1}})/2]$ and $Z_{\frac{1}{2},c}$ is $[(X_{3\cdot 3^{n-1}} + Y_{3\cdot 3^{n-1}})/2, (X_{3\cdot 3^{n-1}} + Y_{3\cdot 3^{n-1}})/2]$, where $Z_{\frac{1}{2},c}$ is the intersection point between the Y = X line and the line which joins the 1/6-way point and 5/6-way point, and $Z_{0,c}$ is the point defined for the 3^n-cascade.

Before the residue R passes the 1/4-resonance value (R=1/2), a pair of stable and unstable orbits of period 4 times the original period are born and at the 1/4-resonance value the newly born orbit is absorbed. After the residue R passes the 1/4-resonance value, the original periodic orbit emits the newly born unstable orbit in such a way that two points which lay on the symmetry line lie off the symmetry line and two points of four points which enclose the 1/2-way point of the original periodic orbit and lay off the symmetry line lie on the symmetry line.

Let us see the structure of the newly born orbit by the resonance of order 4. There are two cases for the $k \cdot 4^n$-cascade. One case is that two different points of the orbits of period $k \cdot 4^n$ lie on the $Y = h(X)$. The other case is that two different points of the orbits of period $k \cdot 4^n$ lie on the $Y = X$. For example, the 4^n-cascade is the former case and the $6 \cdot 4^n$-cascade is the latter case, where the basic orbit of period 6 is the orbit bifurcated from the orbit of period 3. For the 4^n-cascade, let us call the point which lies on the $Y = h(X)$ line the initial point which four points of which two points lie on $Y=h(X)$ enclose.

The intial point Z_0, the 1/4-way point $Z_{4^{n-1}}$, the 1/2-way point $Z_{2 \cdot 4^{n-1}}$ and the 3/4-way point $Z_{3 \cdot 4^{n-1}}$ of the orbit of period 4^n enclose the initial point of the orbit of period 4^{n-1}. The 1/8-way point $Z_{2 \cdot 4^{n-2}}$, the 3/8-way point $Z_{6 \cdot 4^{n-2}}$, the 5/8-way point $Z_{10 \cdot 4^{n-2}}$ and the 7/8-way point $Z_{14 \cdot 4^{n-2}}$ of the orbit of period 4^n enclose the 1/2-way point of the orbit of period 4^{n-1}. By (4), $X_{4^{n-1}} = X_{3 \cdot 4^{n-1}}$. Hence the X-components of 1/4-way point and 3/4-way point are equal. By (4), $X_{2 \cdot 4^{n-2}} = X_{14 \cdot 4^{n-2}}$ and $X_{6 \cdot 4^{n-2}} = X_{10 \cdot 4^{n-2}}$. Hence the X-components of 1/8-way point and 7/8-way point are equal and so are the X-components of 3/8-way point and 5/8-way point. The intersection point between Y=h(X) line and the line which joins the 1/4-way point and the 3/4-way point is $Z_{1,C}$ which is $[X_{4^{n-1}}, h(X_{4^{n-1}})]$, the intersectic point between the Y=h(X) line and the line which joins 1/8-way point and 7/8-way point is $Z_{2,C}$ which is $[X_{2\cdot 4^{n-}}$ $h(X_{2\cdot 4^{n-2}})]$ and the intersection point between the Y=h(X line and the line which joins the 3/8-way point and the 5/8-way point is $Z_{3,C}$ which is $[X_{6\cdot 4^{n-2}}, h(X_{6\cdot 4^{n-2}})]$. For the $6\cdot 4^n$-cascade, let us call the point lying on the Y=X line which four points belonging to the orbit of period $6\cdot 4^{n+1}$ of which two points lie on Y=X enclose the initial point of an orbit of period $6\cdot 4^n$. By (5), $X_{6\cdot 4^{n-1}} = Y_{18\cdot 4^{n-1}}$ and $X_{18\cdot 4^{n-1}} = X_{6\cdot 4^{n-1}}$. Hence the 3/4-way point $Z_{18\cdot 4^{n-1}}$ is the reflection point of the 1/4-way point $Z_{6\cdot 4^{n-1}}$ about the Y=X line. By (5), $X_{3\cdot 4^{n-1}} = Y_{21\cdot 4^{n-1}}$, $X_{21\cdot 4^{n-1}} = Y_{3\cdot 4^{n-1}}$, $X_{9\cdot 4^{n-1}} = Y_{15\cdot 4^{n-1}}$ and $X_{15\cdot 4^{n-1}} = Y_{9\cdot 4^{n-1}}$. Hence the 1/8-way point is the reflection point of the 7/8-way point about the Y=X line and the 3/8-way point is the reflection point of the 5/8-way point about the Y=X line. The intersection poin between the line Y=X and the line which joins 1/4-way point and 3/4-way point is $Z_{1,c}$ which is $[(X_{6\cdot 4^{n-1}} + Y_{6\cdot 4^{n-1}})/2,\ (X_{6\cdot 4^{n-1}} + Y_{6\cdot 4^{n-1}})/2]$, the intersection point between the line Y=X and the line which joins 1/8-way point and 7/8-way point is $Z_{2,c}$ which $[X_{3\cdot 4^{n-1}} + Y_{3\cdot 4^{n-1}})/2,\ (X_{3\cdot 4^{n-1}} + Y_{3\cdot 4^{n-1}})/2]$ and the intersection point between the line Y=X and the line which joins 3/8-way point and 5/8-way point is $Z_{3,c}$ which is $[(X_{9\cdot 4^{n-1}} + Y_{9\cdot 4^{n-1}})/2,\ (X_{9\cdot 4^{n-1}} + Y_{9\cdot 4^{n-1}})/2]$.

Let us define the following sequences for the 3^n-cascade. Like the period-doubling cascade, $\delta'_n \equiv \frac{a_{n-1}-a_n}{a_n-a_{n+1}}$ where a_n is the nonintergrable parameter value at which the orbit of period $k\cdot 3^n$ is unstable.

$\alpha_n(1) \equiv \dfrac{X_0(n) - X_{0,c}(n)}{X_0(n+1) - X_{0,c}(n+1)}$, where $X_0(n)$ is the X-component of the initial point of the orbit of period $k\cdot 3^n$ and $X_{0,c}(n)$ is the X-component of $Z_{0,c}$ of the orbit of period $k\cdot 3^n$ $\beta_n(1) \equiv \dfrac{Y_{1/3}(n) - Y_{2/3}(n)}{Y_{1/3}(n+1) - Y_{2/3}(n+1)}$, where $Y_{1/3}(n)$ is the Y-component of 1/3-way point of the orbit of period $k\cdot 3^n$ and $Y_{2/3}(n)$ is the Y-component of 2/3-way point of the orbit of period $k\cdot 3^n$.

$$\alpha_n(2) \equiv \frac{X_{1/2}(n) - X_{1/2,c}(n)}{X_{1/2}(n+1) - X_{1/2,c}(n+1)}$$

where $X_{\frac{1}{2}}(n)$ is the X-component of the 1/2-way point of the orbit of period $k \cdot 3^n$ and $X_{\frac{1}{2},c}$ is the X-component of $Z_{\frac{1}{2},c}$ of the orbit of period $k \cdot 3^n$,

$$\beta_n(2) \equiv \frac{Y_{1/6}(n) - Y_{5/6}(n)}{Y_{1/6}(n+1) - Y_{5/6}(n+1)},$$ where $Y_{1/6}(n)$ is the Y-component of the 1/6-way point of the orbit of period $k \cdot 3^n$ and $Y_{5/6}(n)$ is the Y-component of the 5/6-way point of the orbit of period $k \cdot 3^n$.

For the 3^n-cascade, it is observed that when n is an even number, both the initial point and the 1/2-way point of newly born orbit move left from the initial point and the 1/2-way point of the original orbit and move right by turns, and for an odd n (≥ 3), the initial point of the newly born orbit moves left from the initial point of the original orbit, as the 1/2-way point of the newly born orbit moves right from the 1/2-way point of the original orbit, and moves right as the 1/2-way point of the newly born orbit moves left by turns, as the non-intergrable parameter a is varied. For the $2 \cdot 3^n$-and the $6 \cdot 3^n$-cascades, it is observed that for an even n, the initial point and the 1/2-way point of the newly born orbit move in such a way that for an odd n, they move for the 3^n-cascade, and for an odd n ($n \geq 1$), they move in such a way that for an even n, they move for the 3^n-cascade. Hence, unlike the period-doubling cascade, the period-trebling α_n-and β_n-sequences converges alternately as $n \to \infty$. In other words, each of the α_n-and β_n-sequences has two different limit values, $\alpha_1, \alpha_2, \beta_1, \beta_2$. From Table 1 to Table 6 it is observed that α_1 is -17.9, α_2 is 2.45, β_1 is 6.02 β_2 is -31.0. On the other hand, like the period-doubling sequence, the δ'_n-sequence converges as $n \to \infty$ and the limit value δ' is 20.18.

Let us define the following new sequences.

$$\delta_n(e) \equiv \frac{a_{2m-2} - a_{2m}}{a_{2m} - a_{2m+2}}, \text{ where } n=2m$$

$$\delta_n(o) \equiv \frac{a_{2m-1} - a_{2m+1}}{a_{2m+1} - a_{2m+3}}, \text{ where } n=2m+1$$

$$\alpha_n(1.e) \equiv \frac{X_0(2m) - X_{0,c}(2m+2)}{X_0(2m+2) - X_{0,c}(2m+2)}, \text{ where } n=2m$$

$$\alpha_n(1,o) \equiv \frac{X_0(2m+1) - X_{0,c}(2m+1)}{X_0(2m+3) - X_{0,c}(2m+3)}, \text{ where } n=2m+1$$

$$\beta_n(1,e) \equiv \frac{Y_{1/3}(2m) - Y_{2/3}(2m)}{Y_{1/3}(2m+2) - Y_{2/3}(2m+2)}, \text{ where } n=2m$$

$$\beta_n(1,o) \equiv \frac{Y_{1/3}(2m+1) - Y_{2/3}(2m+1)}{Y_{1/3}(2m+3) - Y_{2/3}(2m+3)}, \text{ where } n = 2m+1$$

$$\alpha_n(2,e) \equiv \frac{X_{1/2}(2m) - X_{1/2,c}(2m)}{X_{1/2}(2m+2) - X_{1/2,c}(2m+2)}, \text{ where } n=2m$$

$$\alpha_n(2,o) \equiv \frac{X_{1/2}(2m+1) - X_{1/2,c}(2m+1)}{X_{1/2}(2m+3) - X_{1/2,c}(2m+3)}, \text{ where } n=2m+1$$

$$\beta_n(2,e) \equiv \frac{Y_{1/6}(2m) - Y_{5/6}(2m)}{Y_{1/6}(2m+2) - Y_{5/6}(2m+2)}, \text{ where } n=2m$$

$$\beta_n(2,o) \equiv \frac{Y_{1/6}(2m+1) - Y_{5/6}(2m+1)}{Y_{1/6}(2m+3) - Y_{5/6}(2m+3)}, \text{ where } n=2m+1$$

From Table 7 to Table 15 it is observed that $\alpha_n(1,e)$-, $\alpha_n(1,o)$, $\alpha_n(2,e)$ - and $\alpha_n(2,o)$ - sequences converge to the same limit-value $\alpha(=\alpha_1 \cdot \alpha_2)$ which is -44.0, $\beta_n(1,e)$-, $\beta_n(1,o)$- $\beta_n(2,e)$ - and $\beta_n(2,o)$- sequences also converge to the same limit-value $\beta(=\beta_1 \cdot \beta_2)$ which is -187 and $\delta_n(e)$ - and $\delta_n(o)$-sequences also converge to the same limit-value δ which is 408 irrespective of k.

Let us define the following sequences for the 4^n-cascade.

$\delta_n \equiv \dfrac{a_{n-1}-a_n}{a_n-a_{n+1}}$, where a_n is the nonintergrable parameter value at which the orbit of period $k\cdot4^n$ is unstable. $\alpha_n(1) \equiv \dfrac{X_0(n) - X_{1,c}(n)}{X_0(n+1) - X_{1,c}(n+1)}$, where $X_0(n)$ is the X-component of the initial point of the orbit of period $k\cdot4^n$ and $X_{1,c}(n)$ is the X-component of $Z_{1,c}$ of the orbit of period $k\cdot4^n$. $\alpha_n(2) \equiv \dfrac{X_{1/2}(n) - X_{1,c}(n)}{X_{1/2}(n+1) - X_{1,c}(n+1)}$ where $X_{1/2}(n)$ is the X-component of the 1/2-way point of the orbit of period $k\cdot4^n$. $\alpha_n(3) \equiv \dfrac{X_{2,c}(n) - X_{3,c}(n)}{X_{2,c}(n+1) - X_{3,c}(n+1)}$ where $X_{2,c}$ is the X-component of $Z_{2,c}$ of the orbit of period $k\cdot4^n$ and $X_{3,c}$ is the X-component of $Z_{3,c}$ of the orbit of period $k\cdot4^n$. $\beta_n(1) \equiv \dfrac{Y_{1/4}(n) - Y_{3/4}(n)}{Y_{1/4}(n+1) - Y_{3/4}(n+1)}$, where $Y_{1/4}(n)$ is the Y-component of the 1/4-way point of the orbit of period $k\cdot4^n$ and $Y_{3/4}(n)$ is the Y-component of the 3/4-way point of the orbit of period $k\cdot4^n$. $\beta_n(2) \equiv \dfrac{Y_{1/8}(n) - Y_{7/8}(n)}{Y_{1/8}(n+1) - Y_{7/8}(n+1)}$, where $Y_{1/8}(n)$ is the Y-component of the 1/8-way point of the orbit of period $k\cdot4^n$ and $Y_{7/8}(n)$ is the Y-component of the 7/8-way point of the orbit of period $k \cdot 4^n$.

$$\beta_n(3) \equiv \frac{Y_{3/8}(n) - Y_{5/8}(n)}{Y_{3/8}(n+1) - Y_{5/8}(n+1)},$$ where $Y_{3/8}(n)$ is the Y-component of the 3/8-way point of the orbit of the period $k\ 4^n$ and $Y_{5/8}(n)$ is the Y-component of the 5/8-way point of the orbit of period $k \cdot 4^n$

For the 4^n-cascade, it is observed that like the period-doubling bifurcation, the initial point of the newly born orbit moves away from the initial point of the original orbit which moves toward the 1/2-way point of the newly born orbit. It is also observed that before the resonance the initial point and the 1/2-way point of the newly born unstalbe orbit move toward the intial point of the original orbit and are emitted off the symmetry line by the initial point of the original orbit after the resonance, while two points of four points which enclose the 1/2-way point of the original orbit and lay off the symmetry line lie on the symmetry line and one point of the above two points moves away from the 1/2-way point of the original orbit which moves toward the other point.

From Table 16 to Table 21 it is observed that δ_n-sequences converge to the limit-value δ which is 24.5,

$\alpha_n(1)$-, $\alpha_n(2)$- and $\alpha_n(3)$-sequences converge to the limit-value α which is -5.61 and $\beta_n(1)$-, $\beta_n(2)$- and $\beta_n(3)$- sequences converge to the limit-value β which is 14.3 irrespective of k.

Let us calculate δ for the 4^n-cascade by the renormalization schem developed by B. Derrida and Y. Pomeau (the Equality of slope)[17]. The Jacobian matrix M_n of T^n about an orbit of period n is $\prod_{i=1}^{n} M_i$, where $\prod_{i=1}^{n} M_i = \begin{pmatrix} 2h'(X_i)-1 \\ 1 & 0 \end{pmatrix}$ and (X_i, Y_i) is the ith element of the orbit of period n. The eigenvalues of M_n is given by the equation $\lambda_n^2 - \mathrm{Tr}M_n \cdot \lambda_n + 1 = 0$, where λ_n is the eigenvalue of M_n. For the 4^n-cascade, the idea of renormalization is that the linearization of T^n around a point of the orbit of period n is identical to the linearization of T^{4n} around a point of the orbit of period 4n. For the orbit of period 1, $\mathrm{Tr}M_1 = 2-2\sqrt{1+a.}$

For the orbit of period 4 which is born by the 1/4-resonance of the orbit of period 1, $\mathrm{Tr}M_4 = -16a^2-32a^{3/2}+2$. For n=1, $\mathrm{Tr}M_1(a) = \mathrm{Tr}M_4(a')$, where a and a' are the nonintergrable parameter value at which the orbit of period 1 and the orbit of period 4 have the same residue R. Hence, $\sqrt{1+a} - 1 = 8a'^2+16a'^{3/2}-1$

The recursion relation (8) provides an approximate value for a_∞ which is the accumulation point of a_n-

sequence and of δ. a_∞ is the fixed point of the recursion relation (8), whereas δ is given by

$$\delta = da/da'|_{a_\infty} \qquad (9)$$

The fixed point a_∞ is 0.1467 and the numerical value is 0.1427.

Therefore the relative error is 2.8×10^{-2}.

$\delta = da/da'|_{a_\infty})$ is 24.7 and the numerical value is 24.5.

Hence the relative error is 8.2×10^{-3}.

IV. CONCLUSION AND DISCUSSION

From our numerical work, it can be guessed that there exists a universal map under the operation of ninetupling and rescaling, not under the operation of trebling and rescaling for the $k \cdot 3^n$-cascades, and there exists a universal map under the operation of quadrupling and rescaling for the $k \cdot 4^n$-cascades. It seems that the universal rescaling constant δ, α and β are 408, -44.0 and -187 for the $k \cdot 3^n$-cascades and 24.5, -5.61 and 14.3 for the $k \cdot 4^n$-cascade. By the $k \cdot r^n$ cascade, infinite ordinary hyperbolic orbits which are the sources of chaos are born and infinite elliptic orbits which can be the basic orbit of the resonance including the period-doubling bifurcation are born. Hence the discovery of the universal scaling behavior of the period-trebling and the period-guadrupling cascades is of great importance to an understanding of the noninter grable dynamics for which N independent analytic constants of motion do not exist in the dynamic system of N degrees of freedom. From recent numerical works which contain the period-doubling bifurcation studies and our present work how invariant circles with rational winding numbers are destroyed can be understood more deeply than before.

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Table 1

n	a_n	δ'_n
1	1.250000000	20.24608463
2	1.184948799	20.32153602
3	1.181735773	20.18805088
4	1.181577664	20.18782112
5	1.181569832	20.18480928
6	1.181569444	20.18483394
7	1.181569425	
8	1.181569424	

Table 2

n	$\alpha_n(1)$	$\alpha_n(2)$	$\beta_n(1)$	$\beta_n(2)$
2	-17.94807670	2.382765362	6.043232932	-31.65254818
3	2.464334401	-17.96708285	-31.14074300	6.020621327
4	-17.92182852	2.452486664	6.023375256	-31.04739229
5	2.453921972	-17.92745931	-31.03414165	6.016983385
6	-17.89177108	2.453393967	6.017035267	-31.03296551
7	2.458191868	-17.92621298	-31.02931961	6.017112896

Table 1 and Table 2 contain the sequences for the 3^n-cascade.

Table 3

n	a_n	δ'_n
1	3.743333913	20.32963004
2	3.732224438	20.28132436
3	3.731677971	20.19099181
4	3.731651026	20.18703699
5	3.731649692	20.18486982
6	3.731649626	
7	3.731649622	

Table 4

n	$\alpha_n(1)$	$\alpha_n(2)$	$\beta_n(1)$	$\beta_n(2)$
1	-20.41954178	2.633687289	5.308845695	-34.13025554
2	2.424220383	-17.88404758	-31.72010133	6.159911725
3	-17.94229048	2.461078837	6.034140995	-31.10581391
4	2.453147278	-17.92777078	-31.04382625	6.018915472
5	-17.91976143	2.453561405	6.016883488	-31.03476423
6	2.452723605	-17.92626105	-31.03297569	6.017123273

Table 3 and Table 4 contain the sequences for the $2\cdot 3^n$-cascade.

Table 5

n	a_n	δ_n
1	1.273324540	20.23425621
2	1.272975387	20.28140921
3	1.272958132	20.18914267
4	1.272957281	20.18704105
5	1.272957239	
6	1.272957237	

Table 6

n	$\alpha_n(1)$	$\alpha_n(2)$	$\beta_n(1)$	$\beta_n(2)$
1	-20.87389724	2.635949260	5.057408454	-33.56936925
2	2.413444286	-17.87626509	-31.71609119	6.147558217
3	-17.93627187	2.461736108	6.035321745	-31.10168975
4	2.451945540	-17.96267506	-31.04228074	6.019324314
5	-18.23324905	2.434675829	6.017748585	-31.05464845

Table 5 and Table 6 contain the sequences for the $6 \cdot 3^n$-cascade.

Table 7

$\delta_n(e)$	$\delta_n(o)$
410.1247977	411.3758777
407.4844741	407.5496709

Table 8

$\alpha_n(1,e)$	$\alpha_n(1,o)$	$\alpha_n(2,e)$	$\alpha_n(2,o)$
-44.23006285	-44.16537855	-42.81134268	-44.06403109
-43.97876879	-43.90501019	-43.96685488	-43.98312051
-43.98140618		-43.98006276	

Table 9

$\beta_n(1,e)$	$\beta_n(1,o)$	$\beta_n(2,e)$	$\beta_n(2,o)$
-188.1907636	-187.5723808	-190.5680066	-186.9245922
-186.9302809	-186.7335248	-186.8116436	-186.7248379
-186.7045104		-186.7288570	

Table 7, Table 8 and Table 9 contain the sequences for the 3^n-cascade.

TABLE 10

$\delta_n(e)$	$\delta_n(o)$
409.4101992	412.1791339
	407.5904648

TABLE 11

$\alpha_n(1,e)$	$\alpha_n(1,o)$	$\alpha_n(2,e)$	$\alpha_n 2,o)$
-43.49606631	-49.50146940	-44.01405103	-47.10098879
-43.95981398	-44.01508105	-43.98688647	-44.12165727
	-43.95222186		-43.98318223

TABLE 12

$\beta_n(1,e)$	$\beta_n(1,o)$	$\beta_n(2,e)$	$\beta_n(2,o)$
-191.4035638	-168.3971234	-191.6090678	-210.2393613
-186.7870856	-187.3228246	-186.7956226	-187.2232646
	-186.7217990		-186.7400022

Table 10, Table 11 and Table 12 contain the sequences for the $2\cdot 3^n$-cascade.

TABLE 13

$\delta_n(e)$	$\delta_n(o)$
409.3743406	410.3360495

TABLE 14

$\alpha_n(1,e)$	$\alpha_n(1,o)$	$\alpha_n(2,e)$	$\alpha_n(2,o)$
-43.28819285	-50.37798803	-44.00664727	-48.01474101
-44.70693370	-43.97876181	-43.73329080	-44.21936580

TABLE 15

$\beta_n(1,e)$	$\beta_n(1,o)$	$\beta_n(2,e)$	$\beta_n(2,o)$
-191.4168148	-160.4012277	-191.1994484	-206.3696518
-181.8046410	-187.3501520	-186.9280005	-187.2111573

Table 13, Table 14 and Table 15 contain the sequences for the $6\cdot 3^n$-cascade.

Table 16

n	a_n	δ_n
1	0.2174036214	23.43463686
2	0.1459017237	25.02169181
3	0.1428506034	24.45464723
4	0.1427286644	24.47809120
5	0.1427236780	
6	0.1427234743	

Table 17

n	$\alpha_n(1)$	$\alpha_n(2)$	$\alpha_n(3)$
2	-5.487677767	-5.667174864	-4.580167687
3	-5.612448528	-5.566557511	-5.991054023
4	-5.6116719358	-5.630419126	-5.496737200
5	-5.617899899	-5.611580858	-5.641655675

Table 18

n	$\beta_n(1)$	$\beta_n(2)$	$\beta_n(3)$
2	14.60256681	16.93454050	16.08334069
3	14.32370750	13.49097296	13.61123264
4	14.29781191	14.58909657	14.58738808
5	14.27544874	14.20979309	14.21267481

Table 16, Table 17 and Table 18 contain the sequences for the 4^n-cascade.

Table 19

n	a_n	δ_n
1	1.266917429	24.03347034
2	1.266423335	24.96703899
3	1.266402819	24.48424826
4	1.266401997	
5	1.266401963	

Table 20

n	$\alpha_n(1)$	$\alpha_n(2)$	$\alpha_n(3)$
1	-5.982580185	-5.385514669	-6.986122518
2	-5.589441110	-5.790629455	-4.624669760
3	-5.614297500	-5.585121294	-5.928418441
4	-5.612802380	-5.627649640	-5.525441173

Table 21

n	$\beta_n(1)$	$\beta_n(2)$	$\beta_n(3)$
1	14.14456488	12.03661396	14.17862736
2	14.89470455	17.74990498	17.35061375
3	14.36496253	13.66642442	13.75472335
4	14.30044373	14.52217575	14.52593606

Table 19, Table 20 and Table 21 contion the sequences for the 6.4^n-cascade