FINAL REPORT

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The Critical Behaviour in Bidirectionally Coupled Period-Doubling Systems

1. Brief summary of the research work

The main aim of work is the investigation of the FQ-type critical behaviour in bidirectionally coupled period-doubling systems.

1.1. FQ-type critical behaviour in one-dimensional maps

The system of two bidirectionally coupled logistic maps (1) is considered:

$$\begin{aligned} x_{n+1} &= 1 - \lambda x_n^2 - C y_n^2, \\ y_{n+1} &= 1 - A y_n^2 - B x_n^2. \end{aligned} \qquad (1)$$

The values of coupling constants were fixed (B=0.375, C=-0.25) during investigation.

The chart of dynamical regimes in the (λ, A) plane is shown on Fig.1. It is known that besides period-doubling bifurcations there exist Hopf bifurcations. We shall refer to points of intersections of Hopf and period-doubling bifurcations lines as to “period-doubling terminal (PDT) points”. The limit of the PDT point’s sequence is referred as the FQ-point, because in any arbitrary small vicinity of this point both period-doubling and quasiperiodic dynamics exist. This situation was already investigated, and thus parameter values corresponding to FQ-point and scaling constants for parameter and orbit were obtained.

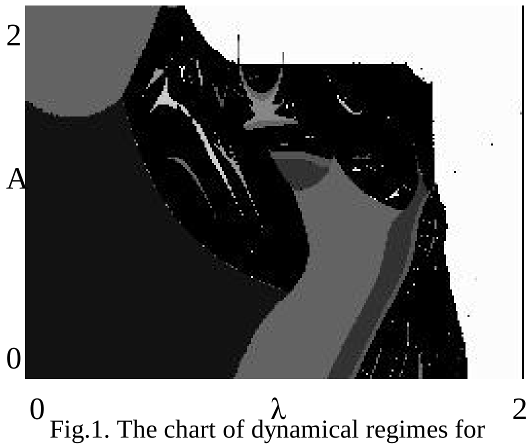

Fig.1. The chart of dynamical regimes for system (1). Different colors corresponds to different periods of cycles, white corresponds to divergence.

In our work we apply two methods to obtain FQ-point and scaling constants.

The first method is to obtain the sequence of PDT points with multipliers (-1,-1). The two-dimensional Newton’s method is used to obtain these points. The obtained points are shown in the Table 1, scaling constants and directions for the parameter – in Table 2 and orbit points and scaling constants for orbit – in Table 3.

Table 1

N	λ	A
1	1.07421785036	0.46629524857
2	1.52325783631	0.903516211061
4	1.62856319614	1.006164376735
8	1.64957035508	1.0261847987
16	1.65336774208	1.0296685493
32	1.65416210948	1.03043362728
64	1.65441685631	1.03071286046
128	1.654496059182	1.030804079503
256	1.654516242274	1.030827673613
512	1.654522026229	1.030834523550
1024	1.654523854245	1.030836706351
2048	1.654524359869	1.030837311944
4096	1.654524500795	1.030837481080
Limit	1.6545245	1.030837

Table 2

N	$\square_1$	x_1/y_1	$\square_2$	x_2/y_2
16	5.191048286254	1.065929975293	-3.969006523789	0.668734555681
32	5.490456160105	1.103370142924	0.933384206011	0.786819955322
64	7.283854233490	0.289330788783	3.293974118756	0.836347811393
128	9.173735755139	0.771392613646	4.452600511128	0.846217825245
256	5.462478138854	0.898722832878	2.724578349685	0.824609631865
512	6.519941546293	0.910688446531	2.853922842411	0.831309510471
1024	7.138633605089	0.815365307846	4.079926778403	0.832261261898
2048	6.065980608273	0.994088565551	3.546091799889	0.831116641483

Table 3

N	x	$\square_1$	y	$\square_2$
1	0.650237539853		0.646523853943	
2	-0.175930635899		-0.178289297324	
4	0.697538014090	-3.362842719076	0.717678115926	-3.298499110225
8	-0.310673108044	-2.436835450487	-0.327438372419	-2.392624283563
16	0.0151969778137	-2.179242677775	0.0165319561600	-2.120953141882
32	-0.00791876805892	-2.001418810928	-0.00883968206162	-1.942160493205
64	0.00414764757688	-1.915709401226	0.00469155470214	-1.875042072250
128	-0.00215392511222	-1.914826065035	-0.00245059275165	-1.894561383857
256	0.00113227067853	-1.917588935760	0.00129202569453	-1.908329036609
512	-0.000598783352278	-1.898378520984	0.000684307203891	-1.893718638698
1024	0.000313763102879	-1.896948940003	0.000358826372366	-1.894611527537
2048	-0.000164765741540	-1.906983175205	-0.000188491303352	-1.905901494918
4096	0.0000870495678226	-1.900316726692	0.0000996005744798	-1.899802520769

One can see that the sequence of PDT points converges to a definite limit which coincides with known values λ_c=1.654524590… A_c=1.030837593…, the sequence of orbit points converges to zero, and scaling factors for both variables converges to a limit value, which coincides with known value □=-1.90007167…. However, the convergence of scaling factors and scaling directions for the parameter is not good, though the values for last level are rather similar to theoretically known values $\square_1$=6.32631925… and $\square_2$=3.44470967….

Second, we apply the eigenvalue-matching method to the investigation of this system. We find the points where cycles of periods N and 2N have equal multipliers. These points and corresponding multipliers are shown in Table 4 and the parameter scaling constants and directions – in Table 5.

Table 4

N	**2N**	**λ**	**A**	**μ_1**	**μ_2**
16	32	1.654485570012	1.030786698827	-1.004819454288	-1.609181163292
32	64	1.654521137159	1.0308334523203	-1.061025528581	-1.572787923814
64	128	1.654527065365	1.030840609566	-1.082683893982	1.575368235694
128	256	1.654524356280	1.030837301540	-1.038502778933	-1.592451193251
256	512	1.654524401496	1.030837407691	-1.049256655802	-1.577039335413
512	1024	1.654524593951	1.030867593199	-1.075773012689	-1.572092585054
1024	2048	1.654524563591	1.030837556541	-1.052624773715	-1.586144945410
2048	4096	1.654524559206	1.030837551269	-1.046615368741	-1.581510536991

Table 5

N	2N	δ_1	x_1/y_1	δ_2	x_2/y_2
16	32	6.277832807093	-1.396701766597	3.302818668512	0.832017219297
32	64	6.409616338481	-0.428745671512	3.393480316995	0.82954334042
64	128	6.371148204283	-3.098849878404	3.496769060739	0.8318427927766
128	256	6.267406434487	-1.133605247685	3.453993604547	0.831689966121
256	512	6.337968433095	0.1575523547884	3.405040322010	0.831007178949
512	1024	6.365158109880	-1.422750637478	3.459879217541	0.8312301237863
1024	2048	6.297144568165	1.356719087031	3.463219942885	0.8313043317477
2048	4096	6.315101753340	0.5119356790225	3.438090557020	0.8311998221856

One can see that in this case practically all characteristics: parameter values, multipliers and scaling constants for the parameters demonstrate rather good convergence to known values (theoretically obtained values for universal multipliers are $\square_1$=-1.057149… and $\square_2$=-1.579739…). However, the first scaling direction for the parameter does not demonstrate any convergence.

So, in this case the eigenvalue-matching method let us to obtain practically all scaling characteristics.

1.2. Bidirectionally coupled Hénon maps.

Then we consider the system of bidirectionally coupled two-dimensional Hénon maps:

$$\begin{aligned} x_{n+1} &= 1-\lambda x_n^2 - by_n - Cu_n^2, y_{n+1} = x_n, \\ u_{n+1} &= 1-Au_n^2 - bv_n - Dx_n^2, v_{n+1} = u_n. \end{aligned} \qquad (2)$$

The values of coupling constants are also fixed as in the case of logistic maps (C=-0.25, D=0.375). Besides it, the damping parameter b is fixed (b=0.2).

Fig.2. The chart of dynamical regimes for system (2).

The chart of dynamical regimes for the system (2) is shown in Fig.2. It seems to be similar to Fig.1, but there are some important differences. In particular, the shape of Arnold's tongues is different in these two cases. In Fig. 3 one can see the Arnold's tongues for Hénon maps (left) and logistic maps (right) in the region, corresponds to Hopf bifurcation of period-2 cycle. For Hénon maps some of tongues forms "rings".

Three methods are applied to obtain the FQ-point in this system and to investigate the scaling behaviour: the "direct" method, the eigenvalue-matching method and the method, based on universal multipliers.

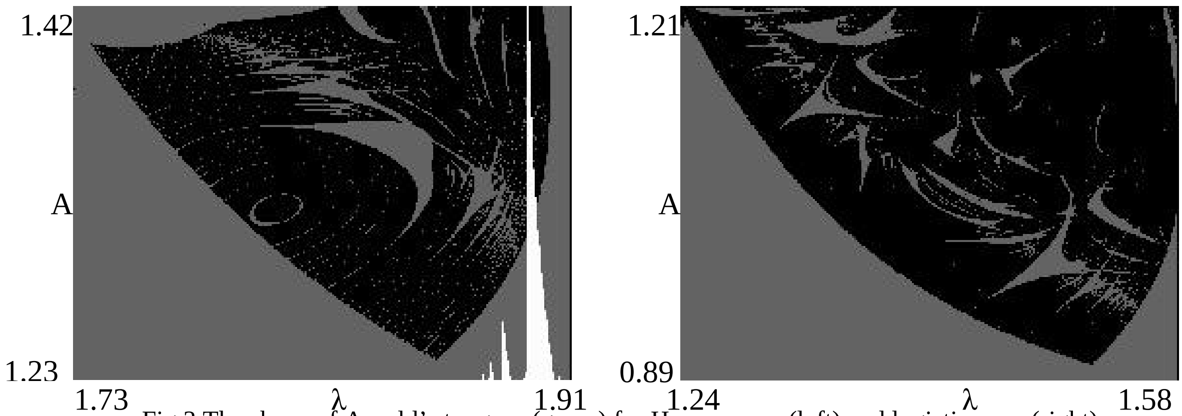

Fig.3 The shape of Arnold's tongues (green) for Henon maps (left) and logistic maps(right).

The results, obtained by the method (i.e. the coordinates of PDT points) are shown in Table 6, and scaling constants for the parameter – in Table 7.

Table 6

N	λ	A
1	1.404435029290	0.796437417357
2	1.861754631834	1.239440665805
4	1.973359033069	1.350745532994
8	1.990560562995	1.366299306704
16	1.996245415954	1.372144428508
32	1.996179915427	1.371830536299
64	1.996707541545	1.372456867033
128	1.996785554422	1.372548574369
256	1.996804748201	1.372571004536
512	1.996804266559	1.372570298860
1024	1.996807235864	1.372573876238
2048	1.996808026958	1.372574830167
Limit(?)	1.99681	1.372575

Table 7

N	□$_1$	x_1/y_1	□$_2$	x_2/y_2
8	4.610604127913	1.023188180363	-1.772540264721	0.815682163532
16	4.731588037864	1.034009825363	-3.115840786182	0.801198434841
32	4.893315446191	1.052707833220	-1.767793314101	0.813614697683
64	5.773516114795	1.061851823799	-1.948422881198	0.791787597517
128	5.992604544375	0.848376369394	-53.676711965007	0.793495298458
256	8.651060741911	0.840214882964	1.672860058992	0.874160492345
512	1.91328192892+ i 2.41263094043	Complex value	1.91328192892- i 2.41263094043	Complex value
1024	7.851900932442	0.8678952769583	-0.965509401758	0.826407999157
2048	35.839280517512	0.8234232060616	4.211473087502	0.829215337920

One can see that sequence of the PDT points demonstrates anomalous behaviour for some points: for periods 32 and 512 the corresponding parameter values are less than previous values. The graphical illustration of this fact for period 32 is shown in Fig. 5. Also one can see that the shape of period-16 region is different from that in the case of logistic map (Fig.4).

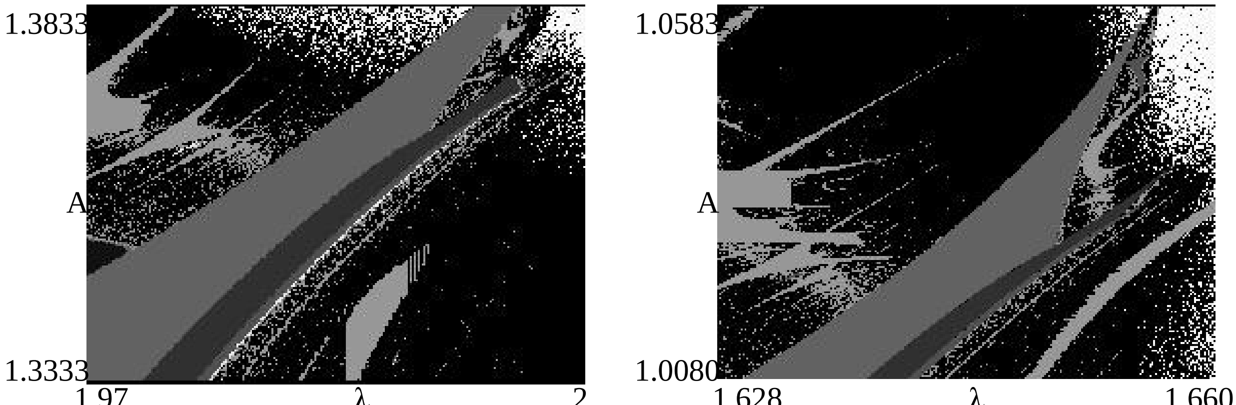

Fig. 4. The region of period 16 for Henon maps (left) and logistic maps (right). Period 8 is marked with green, period 16 – red, 32 – grey, 64 – yellow, other periods – pink, chaos and quasiperiodicity –black.

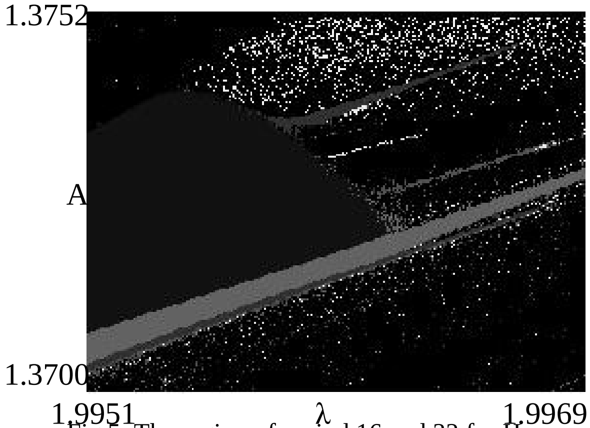

Fig.5. The region of period 16 and 32 for Henon maps. One can see PDT points for these periods. Period 16 marked with blue, 32 – green, 64 –red, 128 – grey, 256- yellow.

One can see that though the sequence of the PDT points demonstrates convergence, no convergence may be seen in the scaling constants.

Then we applied the eigenvalue-matching method to investigate this system. It should be marked, that, rigorously speaking, it is impossible to apply eigenvalue-matching method in this case, because the determinant of stability matrix for period-N cycle is constant and equal b^{2N}, so it is impossible to match all eigenvalues. But for large N determinant becomes less then numeric accuracy, so the method can be applied in this case. For b=0.2 period N=16 is enough to make determinant less than the numeric accuracy.

The parameter values obtained by this method and multipliers in these points are shown in Table 8.

Table 8

N	2N	λ	A	μ_1	μ_2
32	64	1.996912439816	1.372704166059	-1.558540831721+ i 0.36070988907	-1.558540831721- i 0.36070988907
64	128	1.996800414518	1.372565542832	-1.630211655524	-0.995236942232
128	256	1.996813488520	1.372581395896	-1.687223965071	-1.179132048730
256	512	1.996802101793	1.372567629176	-1.585239013244	-0.574712641268
512	1024	1.996808384299	1.372575262920	-1.482352609659+ i 0.17738320550	-1.482352609659- i 0.17738320550

One can see that the sequence of parameters seems to converge but there is no convergence for multipliers and scaling factors (the Table 9). Note also that complex multipliers correspond to the periods of anomalous behaviour of PDT-point sequence.

There exist two possibilities to obtain orbit scaling factor: from the sequence of cycle's initial elements and from the sequence of maximum distances between the cycle's elements. The orbit points are shown in Table 10, the scaling factors obtained from this data – in Table 11, the maximum distances – in Table 12 and the scaling factors obtained from this data – in Table 13.

Table 9

N	2N	$\square_1$	x_1/y_1	$\square_2$	x_2/y_2
32	64	10.922514659439	0.619983043430	3.572088344194	0.822044072145
64	128	3.839533154996+ I 0.449044669154	Complex value	3.839533154996- I 0.449044669154	Complex value
128	256	10.496833297409	0.769194209648	3.687652371341	0.837312325281
256	512	4.8531849223206	0.869884884806	1.806063087271	0.819519817697
512	1024	8.9414001109635	0.804632996623	3.671437889006	0.827040427495

Table 10

N	2N	x	Y	u	v
32	64	0.026716677798	0.762047055351	0.025689815092	0.762255927148
64	128	0.037991626650	0.758127373551	0.038263375768	0.757867003481
128	256	0.045139388621	0.757258705206	0.041060315369	0.756873396327
256	512	0.039177109851	0.757710017140	0.039606746817	0.757391377410
512	1024	0.039841646418	0.757474545041	0.040306668000	0.757120427527

Table 11

N	2N	$\square_1$	$\square_2$	$\square_3$	$\square_4$
128	256	4.586017208439	4.512288060854	4.495470932846	4.417162371403
256	512	-1.929528399457	-1.924762629143	-1.924188299369	-1.918230580582
512	1024	-1.917552921613	-1.916626022845	-1.912758371365	-1.911722850490

Table 12

N	2N	dx	dy	du	dv
32	64	0.016730423580	0.0058643159778	0.018764973266	0.0066110597547
64	128	0.007101007401	0.0025008549090	0.008015207202	0.0028390418062
128	256	0.003731098823	0.001313074866	0.004241387801	0.0015011301670
256	512	0.001572039767	0.0005538902748	0.001791367934	0.0006348586092
512	1024	0.001151538423	0.0004057919256	0.001317294210	0.0004669263501

Table 13

N	2N	$\square_1$	$\square_2$	$\square_3$	$\square_4$
64	128	2.356063391419	2.344924512257	2.341171325053	2.328623601154
128	256	1.903194671941	1.904579070540	1.889760516836	1.891269570468
256	512	2.373412493013	2.370640768083	2.367680988356	2.364511003259
512	1024	1.365164840082	1.364961301364	1.359884466914	1.359654706038

The sequence of orbit points seems to converge. The scaling constants' sequences demonstrate rather bad convergence, but it is interesting that for scaling constant obtained from maximum distance the value for period 256 is close to theoretically known, but then the sequence seems to diverge.

The third possible way to obtain the FQ-point is to find the points where multipliers are equal to the theoretically known universal values. We follow the sequence of such points up to period 1024 (see Table 14). This sequence seems to converge rather well, but it seems to be very strange that we fail to obtain this point for period 2048 even when using PDT-point as initial condition (because for the PDT point both multipliers equal -1 and it seems to be good initial condition to obtain point with universal multipliers). The results for orbit scaling in this method are practically the same that in the previous case.

Also we consider the last term of this sequence (for period 1024) as an approximation of possible critical point and calculate the multipliers and orbit points for cycles of different periods in this point (see Tables 15 and 16, accordingly). The scaling constants for orbit calculated from cycle's elements are shown in Table 17, and calculated from maximum distance – in Table 19. The values for maximum distances are shown in Table 18.

Table 14

N	λ	A
16	1.997357158325	1.373290976724
32	1.996481416415	1.372156285497
64	1.996807358040	1.372574647929
128	1.996803848836	1.372569864294
256	1.996810908701	1.372578313345
512	1.996806179926	1.372572588256
1024	1.996807846694	1.372574612036

Table 15

N	$\square_1$	$\square_2$
16	-1.631708696357	-0.794099439267
32	-1.489549775563+i 0.281788876158	-1.489549775563-i 0.281788876158
64	-1.638614715859	-1.020365440080
128	-1.641740866200	-1.101527247477
256	-1.585269117854	-0.859324215727
512	-1.433937671094+i 0.279856292249	-1.433937671094-i 0.279856292249
1024	-1.579739494183	-1.057148718750

Table 16

N	x	y	u	v
16	0.051054973839	0.753561262544	0.052033243046	0.753038200646
32	0.032168164678	0.760164196056	0.031797030925	0.760139370286
64	0.042923887951	0.756389123534	0.043848386578	0.755887276929
128	0.037925655421	0.758149117398	0.038187425068	0.757892079828
256	0.040342277466	0.757298316321	0.040935657672	0.756918986078
512	0.039065317482	0.757748261552	0.039478476435	0.757435439467
1024	0.039811472345	0.757485293893	0.040332087383	0.757132830827

Table 17

N	$\square_1$	$\square_2$	$\square_3$	$\square_4$
64	-1.755977602028	-1.749087858185	-1.679164793047	-1.670040858419
128	-2.151905340226	-2.144935047341	-2.128853134174	-2.120953316219
256	-2.068272339211	-2.068631448147	-2.0598553054645	-2.060236127300
512	-1.8924806378271	-1.890899199240	-1.8859923077639	-1.884185041140
1024	-1.711387337028	-1.711028773314	-1.7070788986647	-1.7066709959107

Table 18

N	dx	dy	du	dv
16	0.059300729672	0.020397432885	0.061426275817	0.021168670926
32	0.027294454695	0.009558239654	0.029079890649	0.010220544610
64	0.015821794530	0.005544765210	0.017725139675	0.006243283812
128	0.007225615000	0.002544812059	0.008157667832	0.002889597738
256	0.003525998285	0.001240851844	0.004007368501	0.001418248781
512	0.001824355487	0.000642883820	0.002080849139	0.000737572883
1024	0.001098494398	0.000387095042	0.001256529232	0.000445381559

Table 19

N	$\square_1$	$\square_2$	$\square_3$	$\square_4$
32	2.172629214786	2.134015637122	2.112328294436	2.071188154229
64	1.725117504417	1.723831270035	1.640601494950	1.637046291305
128	2.189681366914	2.178850571849	2.172819492045	2.160606554295
256	2.049239510619	2.050858908987	2.035667004411	2.037440663945
512	1.932736415751	1.930133883289	1.925833269646	1.922859169160
1024	1.660778143540	1.660790633428	1.656029231957	1.656047198398

We can see that though for one value of period scaling constants seems to be close to theoretically known, no clear tendency to converge can be observed both for multipliers and orbit scaling constants. Note also, that complex multipliers correspond to points of anomalous behaviour of the PDT-point sequence. On the whole, this sequence these data demonstrate the same properties than data obtained from eigenvalue-matching method.

2. Brief explanation of the results obtained during the visit period.

The results of investigation of system of bidirectionally coupled 2D (Hénon) maps let us to draw next conclusion: in spite of similarity its global parameter plane construction to that of coupled 1D maps, the behaviour of this system near the FQ-point seems to be qualitatively different. In particular, the FQ-point in this system seems to be “intermediate asymptotic” (similar to tricriticality case), i.e. the sequence of parameter seems to demonstrate convergence, but there is no scaling in high levels. However, this is only suggestion, and further investigations should be carried to confirm or refute it.

3. Future plan to complete the work and publication plan.

The first aim of future work is to confirm (if it is possible) the fact, that the FQ-point exists as “intermediate asymptotic”. It may be very useful for this purpose to obtain points with universal multipliers for next few levels (may be, by increasing the value of b from 0 to 0.2). Also it is necessary to try to obtain the FQ-point for other values of b, both smaller and larger then 0.2.

The other direction is the investigation of the structure in the parameter plane of the coupled Hénon maps in comparison with that of coupled logistic maps. In particular, it seems to be interesting to calculate the rotation numbers of Arnold’s tongues for both cases. Also the changes in structure occur when varying coupling parameters are to be studied.

After that, analogous investigations for coupled oscillators are planned to be made.

As for publications, it seems to be possible to publish the results of the investigation of bidirectionally coupled Hénon maps when this work will be completed. As it requires more then one paper, another one may be published when the investigations of bidirectionally coupled oscillators will be completed.